Solution 9

Supplementary Problems

1. Verify Green's theorem when the region D is the rectangle $[0,a]\times[0,b]$.

   Solution. The boundary of the rectangle consists of four curves: $C_1, x\mapsto(x,0), x\in[0,a]; C_2, y\mapsto(a,y), y\in[0,b]; C_3, x\mapsto(x,b), x\in[0,a]; C_4, y\mapsto(0,y), y\in[0,b]$ and $C=C_1+C_2-C_3-C_4$. We have

$$\int_{C_1} Mdx+Ndy=\int_0^a M(x,0)dx,$$

$$\int_{C_2} Mdx+Ndy=\int_0^b N(a,y)\,dy\ ,$$

$$\int_{C_3} Mdx+Ndy=\int_0^a M(x,b)\,dx\ ,$$

$$\int_{C_4} Mdx+Ndy=\int_0^b N(0,y)\,dy\ .$$

   It follows that

$$\begin{aligned}\int_C Mdx+Ndy &= \left(\int_{C_1}+\int_{C_2}-\int_{C_3}-\int_{C_4}\right)Mdx+Ndy\\ &= \int_0^a M(x,0)dx+\int_0^b N(a,y)\,dy-\int_0^a M(x,b)\,dx-\int_0^b N(0,y)\,dy\ .\end{aligned}$$

   On the other hand,

$$\begin{aligned}\iint_D (N_x-M_y)\,dA &= \iint_D N_x dA-\iint_D M_y dA\\ &= \int_0^b\int_0^a N_x dxdy-\int_0^a\int_0^b M_y dydx\\ &= \int_0^b N(a,y)dy-\int_0^b N(0,y)\,dy-\int_0^a M(x,b)dx+\int_0^a M(x,0)\,dy\ .\end{aligned}$$

   By comparing these two formulas, we conclude

$$\int_C Mdx+Ndy=\iint_D (N_x-M_y)\,dA\ .$$

2. Let D be the parallelogram formed by the lines $x+y=1, x+y=3, y=2x-3, y=2x+2$. Evaluate the line integral

$$\oint_C dx+3xy\,dy$$

   where C is the boundary of D oriented in anticlockwise direction. Suggestion: Try Green's theorem and then apply change of variables formula.

   Solution. By Green's theorem

$$\oint_C dx+3xy\,dy=\iint_D 3y\,dA(x,y)\ .$$

Next, let $u = x + y$ and $v = y - 2x$. Then $(u, v) \mapsto (x, y)$ sends the rectangle $R = [1, 3] \times [-3, 2]$ to D. We have $\dfrac{\partial(u,v)}{\partial(x,y)} = 3$ and $x = (u - v)/3$ and $y = (2u + v)/3$. By the change of variables formula

$$\begin{aligned}\iint_D 3y\,dA(x,y) &= \iint_R (2u+v)\frac{1}{3}\,dA(u,v)\\ &= \frac{1}{3}\int_1^3\int_{-3}^2 (2u+v)\,dvdu\\ &= \frac{1}{3}\int_1^3 (10u-5)\,du\\ &= \frac{35}{3}\ .\end{aligned}$$

3. Find a potential for the vector field

$$\frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}\ ,$$

in the region obtained by deleting the line $(x, 0), x \le 0$, from $\mathbb{R}^2$.

Solution. From $\dfrac{\partial \Phi}{\partial y} = \dfrac{x}{x^2+y^2}$, etc we get

$$\Phi(x,y) = \tan^{-1}\frac{y}{x}\ .$$

This is the argument, that is, the angle between (x, y) and the positive x-axis.

If you start with $\dfrac{\partial \Phi}{\partial x} = \dfrac{-y}{x^2+y^2}$, you get

$$\Phi(x,y) = -\tan^{-1}\frac{x}{y}\ ,$$

which is the same as the first one after observing the relation $\tan(\pi/2 - \theta) = -1/\tan\theta$.

4. Let $F = M\mathbf{i} + N\mathbf{j}$ be a smooth vector field in $\mathbb{R}^2$ except at the origin. Suppose that $M_y = N_x$. Show that for any simple closed curve γ enclosing the origin and oriented in anticlockwise direction, one has

$$\oint_\gamma Mdx + Ndy = \varepsilon\int_0^{2\pi}\left[-M(\varepsilon\cos\theta,\varepsilon\sin\theta)\sin\theta + N(\varepsilon\cos\theta,\varepsilon\sin\theta)\cos\theta\right]d\theta\ ,$$

for all sufficiently small ε. What happens when γ does not enclose the origin?

Solution. Let γ_ε be the circle entered at the origin with radius ε which is so small to be enclosed by γ. Then the vector field $\mathbf{F}$ is smooth in the region bounded by γ and γ_1. Applying Green's theorem in a multi-connected region we have

$$\oint_\gamma Mdx + Ndy = \oint_{\gamma'} Mdx + Ndy\ .$$

Using the standard parametrization, $\theta \mapsto (\varepsilon\cos\theta, \varepsilon\sin\theta)$, we further have

$$\oint_{\gamma'} Mdx + Ndy\ = \varepsilon\int_0^{2\pi}\left[-M(\varepsilon\cos\theta,\varepsilon\sin\theta)\sin\theta + N(\varepsilon\cos\theta,\varepsilon\sin\theta)\cos\theta\right]d\theta\ ,$$

for all sufficiently small ε.

The line integral vanishes when γ does not include the origin.